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Geometric Stiffening in Multibody Dynamics Formulations

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Controversy over the issue of geometric stiffening as it arises in the context of multibody dynamics revolves primarily around the "correct" methodology for incorporating the stiffening effect into dynamics formulations. The main goal of this work is to present the different approaches that have been developed for this problem through an in-depth review of several publications. The contribution is a precise understanding of the existing methods and how they relate to each other. The paper also offers some novel insights and clarifying interpretations. It concludes with a general classification and a numerical comparison of the approaches for modeling geometric stiffening in flexible-body systems.

I. Introduction

THE issue of geometric stiffening, also referred to as dynamic, centrifugal, or rotational stiffening and foreshortening, has been a topic of many recent publications dealing with the dynamics of flexible bodies for applications to multibody systems. Kane et al.[1] were the first to observe that the majority of existing multibody dynamics formulations and, accordingly, the simulation software do not incorporate the geometric stiffening effect. They attributed this flaw to the conventional approach for describing the deformation of elastic bodies, which yields a set of dynamics equations that inherently lack the stiffening terms. Eke and Laskin[2] qualified the error in the conventional approach as a premature linearization of the displacement field.

Shortly after Kane et al.'s[1] publication, two commentaries appeared on the material in Ref. 1. In particular, London[3] pointed out that several approaches have been previously employed to include the geometric stiffening effect in the dynamics equations and compiled a table characterizing the various methods. In a technical note, Hanagud and Sarkar[4] state that the conventional method for modeling the kinematics of elastic deformation can be used.

The controversy over the nature of geometric stiffening, the debate on the "correct" approach to model it, and the seeming incongruity of the existing methods—all of these motivated the author to review several of the works on this subject. In doing so, an attempt was made to understand precisely how geometric stiffening is incorporated into the dynamics equations in different approaches, what assumptions and approximations are made in the derivation, and whether they are justified. This paper contains the main results of the review.

The starting point of the presentation will be the landmark paper by Kane et al.[1] and the commentary.[4] Following that, a thorough treatment of the works by Likins et al.,[5] Vigneron,[6] and Kaza and Kvaternik[7] is given. Section IV contains the main results from Laskin et al.[8] and the recent works by Simo and Vu-Quoc,[9] Meirovitch,[10] and Banerjee et al.[11,12] In reviewing these works, we do not simply repeat the derivations, or include the dynamics equations developed in these publications. Instead, the focus is on the fundamental assumptions made in formulating these equations, where the formulation ends when the development becomes a purely mechanical process. In addition to presenting the key features of different procedures and establishing relationships between them, the author attempts to disclose some of the existing misconceptions. A classification of the approaches and comments on their suitability for multibody dynamics simulation culminate the review. The paper is completed with a numerical comparison based on a spin-up of a flexible-beam problem.

II. Kane et al. and Commentaries

Main Results of Kane et al.

In Ref. 1, Kane et al. develop the dynamics equations of a general flexible beam built into a rigid body undergoing arbitrary but prescribed motion. Their formulation differs from many existing procedures in several respects. First, it incorporates the effect of the transverse displacement on the axial displacement in the kinematic description of the deformation. This is achieved indirectly by expressing the distance along the deformed elastic axis as a nonlinear function of the transverse displacements with

$$x + s(x, t) = \int_0^{x+u_1} \left[1 + \left(\frac{\partial u_2}{\partial \sigma} \right)^2 + \left(\frac{\partial u_3}{\partial \sigma} \right)^2 \right]^{\frac{1}{2}} d\sigma \qquad (1)$$

The above equation is the same as Eq. (19) of Ref. 1, where the variable s denotes "the stretch in the beam along the elastic axis."

The second major difference relates to the choice of variables employed to describe kinematics of the deformed beam. In particular, Kane et al. employ the stretch s with the transverse translations u_2 and u_3 as a set of generalized coordinates. Thus, they discretize the continuous displacement field as follows:

$$\begin{aligned} s(x, t) &= \sum_{j=1}^{\nu} \phi_{1j}(x) q_j(t) \\ u_i(x, t) &= \sum_{j=1}^{\nu} \phi_{ij}(x) q_j(t), \qquad i = 2, 3 \end{aligned} \qquad (2)$$

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By contrast the conventional approach involves discretizing the orthogonal set of elastic displacements $\{u_1, u_2, u_3\}$ with

$$u_i = \sum_{j=1}^{\nu} \phi_{ij}(x)\, q_j(t), \qquad i = 1, 2, 3 \tag{3}$$

Kane et al. argue that the standard procedure cannot account for the fact that every transverse displacement gives rise to an axial displacement.

The general methodology employed to derive an explicit (literal) set of motion equations for the elastic coordinates follows that presented in Kane and Levinson.[13] The procedure requires one to construct the generalized inertia and generalized active forces. The former are developed according to the algorithm in Ref. 13. The generalized active forces, which for the system considered result from internal forces, are derived from the strain energy function U. It is initially written in terms of the squares of internal loads, each of which is expressed as a linear function of elastic deformations or their spatial derivatives. For reasons that will become apparent in the Commentaries section, we draw attention to one particular term in the strain energy [the first term in Eq. (60) of Ref. 1]. It represents the contribution of the axial load and is rewritten here by omitting the subscripts as

$$U_P = \frac{1}{2}\int_0^L EA\left(\frac{\partial s}{\partial x}\right)^2 \mathrm{d}x \tag{4}$$

The equations of motion presented in Ref. 1 are linear in the elastic coordinates, and they incorporate the geometric stiffening. In fact, one of the crucial points made by these authors is that the linear dynamics equations in other works do not contain all of the linear terms, in particular the stiffening term. This will be further discussed in the Commentaries.

Hanagud and Sarkar's Note

Contrary to that claimed in Ref. 1, Hanagud and Sarkar[4] believe that the axial and transverse motions can be treated independently with the standard discretization procedure and the stiffening effect can be accounted for if "the nonlinear effects are properly included in the formulation."[4]

Hanagud and Sarkar present a formulation where, as in the conventional approach, they discretize the axial displacement u_1, not the stretch variable s. The differential equations for the corresponding discrete coordinates are derived by using the same general methodology as in Ref. 1. Hanagud and Sarkar also determine the generalized elastic forces from the strain energy function. However, they formulate U as a quartic function of the spatial derivatives of u_1, u_2, and u_3. This is accomplished by using the nonlinear strain-displacement relations, through which the aforementioned nonlinear effects are introduced. Lastly, Hanagud and Sarkar do not linearize the final equations of motion, but retain terms of second and third order.

A notable observation made by Hanagud and Sarkar is that the expression for the stretch variable in Ref. 1 is inconsistent with the rest of their development. The inconsistency results from the fact that Eq. (1) is applicable if one expresses the transverse displacements u_2 and u_3 as a function of the deformed coordinate, whereas Kane et al.[1] express these in terms of the undeformed coordinate x. The consistent expression for the stretch variable is

$$x + s(x,t) = \int_0^x \left[\left(1 + \left.\frac{\partial u_1(x,t)}{\partial x}\right|_{x=\sigma}\right)^2 + \left(\left.\frac{\partial u_2(x,t)}{\partial x}\right|_{x=\sigma}\right)^2 + \left(\left.\frac{\partial u_3(x,t)}{\partial x}\right|_{x=\sigma}\right)^2\right]^{\frac{1}{2}} \mathrm{d}\sigma \tag{5}$$

which is equivalent to Eq. (2) in Ref. 4. It is observed that Eq. (5) embodies the nonlinear formulation of the strain.

Commentaries

In the following, the author offers additional clarification on the formulation proposed by Kane et al.[1] In particular, London[3] questions why the foreshortening is not considered during evaluation of the strain energy in Ref. 1. The analysis below shows that the "axial" strain energy, cited in Eq. (4), does comprise the foreshortening effect.

It will be helpful to introduce the axial component of the Lagrangian strain tensor, denoted by $\varepsilon_{0,xx}$. (The 0 subscript is used to refer to the elastic axis.) The strain $\varepsilon_{0,xx}$ can be expressed in terms of the elastic displacements with a well-established nonlinear strain-displacement relation

$$\varepsilon_{0,xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_3}{\partial x}\right)^2\right] \tag{6}$$

Then, the stretch gradient derived from the consistent expression (5) can be succinctly written as

$$\frac{\partial s}{\partial x} = [1 + 2\varepsilon_{0,xx}]^{\frac{1}{2}} - 1 \tag{7}$$

The above can be simplified by using the binomial theorem, combined with the small strain assumption, to give

$$\frac{\partial s}{\partial x} \approx [1 + \varepsilon_{0,xx}] - 1 = \varepsilon_{0,xx} \tag{8}$$

Substituting for $\partial s/\partial x$ from Eq. (8), the axial contribution to the strain energy function employed by Kane et al.[1] takes the form

$$U_P = \frac{1}{2}\int_0^L EA(\varepsilon_{0,xx})^2\, \mathrm{d}x \tag{9}$$

with $\varepsilon_{0,xx}$ given by the nonlinear expression (6).

It is now appropriate to consider the form of the strain energy employed by Hanagud and Sarkar in their formulation [Eq. (8) in Ref. 4]. This expression can be derived by using Eq. (9) in conjunction with the approximate axial strain:

$$\varepsilon_{0,xx} \approx \frac{\partial u_1}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_3}{\partial x}\right)^2\right] \tag{10}$$

The third- and fourth-order terms in the strain energy of Ref. 4 result from the nonlinear terms in Eq. (10). It is these terms that lead to foreshortening and stiffening of the beam.

The above development demonstrates that the geometric nonlinearity is included in the strain energy given in Ref. 1. This is not apparent because Kane et al.[1] employ the stretch s instead of the axial translation u_1 as a generalized coordinate in their formulation. As a result, their strain energy remains quadratic in form, which is precisely the reason why the final equations of Ref. 1 are linear in elastic coordinates, but yet include the stiffening term. As the above analysis shows, it is through the use of stretch instead of axial displacement, that Kane et al. incorporate foreshortening in their formulation, without introducing nonlinearity explicitly into the motion equations. By contrast, Hanagud and Sarkar obtain the stiffening effect with the conventional approach by employing the nonlinear strain-displacement relations to construct the strain energy function. In this case, the resulting stiffening term is nonlinear function in elastic coordinates.

III. Likins et al., Vigneron, and Kaza and Kvaternik

The choice to discuss the contributions of Likins et al.,[5] Vigneron,[6] and Kaza and Kvaternik[7] in the same section is based on several reasons. First, these articles appeared within a time span of four years and therefore belong to the same era. Second, all three publications contain formulations of the dynamics equations for a flexible beam spinning in a plane at a constant speed. There is also a logical relationship between these works, since Vigneron's is a comment on Ref. 5, whereas Kaza and Kvaternik's is an extension of Vigneron's approach. The results of the three works are presented in a common notation, which is also employed throughout the rest of the paper. In this notation, the three orthogonal elastic displacements of any point in the beam are denoted by u, v, and w, as in Kaza and Kvaternik.[7]

As already mentioned, the system considered in all three publications is an Euler–Bernoulli beam with a symmetric cross-section, spinning at a constant angular speed Ω about the z axis of the reference frame. (Likins et al.,[5] in addition, consider axial beams.) The common features of the formulations[5–7] are listed below:

1) The dynamics equations are constructed via Hamilton's principle.

2) The position of a generic point located at $\boldsymbol{r}^T = [x, y, z]^T$ in the undeformed beam is given by $[x + u,\ y + v,\ z + w]^T$, where $v(x, y, z, t) = v_0(x, t)$ and $w(x, y, z, t) = w_0(x, t)$. The latter implies a linearization of the displacement field and precludes torsional deformation.

3) The kinetic energy is calculated according to

$$T = \frac{1}{2}\int\int\int[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \Omega^2(x+u)^2 + \Omega^2(y+v)^2 + 2\Omega(x+u)\dot{v} - 2\Omega(y+v)\dot{u}]\frac{\rho}{A}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{11}$$

where ρ denotes the mass per unit length of the beam and A is the crosssectional area.

4) The potential energy is calculated with

$$U = \frac{E}{2}\int\int\int(\varepsilon_{xx})^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{12}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{13}$$

5) The dynamics equations are formulated for the continuous displacement variables.

There are two major differences between formulations presented in Refs. 5–7. The first one relates to the form of the assumed axial displacement field. Likins et al. "expand" u with

$$u = u_0 - y\frac{\partial v}{\partial x} - z\frac{\partial w}{\partial x} \tag{14}$$

while Vigneron, followed by Kaza and Kvaternik, adopts a different form. Their axial displacement is given by Eqs. (2) and (1a) in Refs. 6 and 7, respectively, which we write as

$$u = u_s - y\frac{\partial v}{\partial x} - z\frac{\partial w}{\partial x} - u_f \tag{15}$$

In the above, u_f is the "displacement associated with the foreshortening effect,"[6] whereas u_s results strictly from extension or stretch of the elastic axis. In both references, u_f is specified as an explicit function of the transverse displacements

$$u_f = u_f(x, t) = \frac{1}{2}\int_0^x\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x \tag{16}$$

The second difference between the derivations in Refs. 5–7 lies in the approximations made in the kinetic and strain energies of the system. Likins et al. substitute their expansion for u into expressions (11) and (12), the latter combined with (13). They simplify the result by: (a) assuming that the vibration of the beam takes place in the meriodinal direction only; (b) neglecting all terms that involve u_0 and $\dot{u}_0$ in T, thus eliminating the axial equation of motion from the model; (c) retaining only the third-order term in the strain energy U among the additional terms arising from the nonlinearities in ε_{xx}. With the above assumptions, the kinetic and potential energy functions become

$$T = \frac{1}{2}\rho\int_0^L\dot{w}^2\,\mathrm{d}x + \rho\Omega^2L\left(\frac{1}{2A}I_z + \frac{1}{6}L^2\right) \tag{17}$$

$$U = \frac{EI_z}{2}\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x + \frac{EA}{2}\int_0^L\frac{\partial u_0}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \tag{18}$$

Note that the second (third-order) term in (18) represents the coupling between axial and transverse displacements that leads to the stiffening of the beam in bending. It is identical to the corresponding third-order term in the strain energy function of Hanagud and Sarkar.[4]

At this point in their development, Likins et al. establish a connection between their approach and "the textbook derivation for the transverse vibrations of beams subject to an external axial force P." As noted by them, the axial load to first approximation is given by $P = EA(\partial u_0/\partial x)$, so that Eq. (18) can be rewritten as

$$U = \frac{EI_z}{2}\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x + \frac{1}{2}\int_0^L P\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \tag{19}$$

Furthermore, for the problem considered, P is the centrifugal load and is calculated with

$$P(x) = \tfrac{1}{2}\rho\Omega^2(L^2 - x^2) \tag{20}$$

With Eqs. (17), (19), and (20), Likins et al. derive the motion equation for the transverse displacement w. Due to the assumptions made in evaluating the strain energy and the axial load, geometric stiffening appears as a linear term in this equation.

Let us now proceed with the developments presented by Vigneron, and Kaza and Kvaternik. They construct the kinetic energy from Eq. (11) with (15), ignoring the effects of rotary inertia. The result, as a function of u_s, v, w, and u_f, is

$$\begin{aligned} T = {} & \frac{1}{2}\rho\int_0^L\left(\dot{u}_s^2 + \dot{v}^2 + \dot{w}^2 - \underline{2\dot{u}_s\dot{u}_f} + \underline{\underline{\dot{u}_f^2}}\right)\mathrm{d}x \\ & + \rho\Omega\int_0^L(-\dot{u}_s v + v\dot{u}_f + x\dot{v} + \dot{v}u_s - \underline{u_f\dot{v}})\,\mathrm{d}x \\ & + \frac{\Omega^2}{2}\rho\int_0^L\left(x^2 + u_s^2 + v^2 + \underline{\underline{u_f^2}} + 2xu_s - [2xu_f] - \underline{2u_su_f}\right)\mathrm{d}x \end{aligned} \tag{21}$$

and it includes terms of third and fourth order. These are underlined in Eq. (21) with single and double lines, respectively, whereas the term in square brackets is linear in u_f. This term gives rise to the stiffening effect in the transverse motion equations. It should be emphasized that the higher order terms in Eq. (21) are a consequence of dividing the axial displacement u_0 into u_s and u_f with a second-order form for the latter.

Vigneron further approximates T by keeping only second-order terms in Eq. (21) as well as setting the axial displacement u_s and its derivatives to zero. The latter approximation corresponds to the inextensibility assumption, which implies that the beam is modeled as axially rigid.[7] Kaza and Kvaternik retain third-order terms in Eq. (21) and do not assume inextensibility.

To determine the strain energy, Vigneron and Kaza and Kvaternik first evaluate the strain ε_{xx} of Eq. (13) by substituting for the axial displacement from Eq. (15) in conjunction with Eq. (16). This produces the following second-order approximation:

$$\varepsilon_{xx} \approx \frac{\partial[u_s - y(\partial v/\partial x) - z(\partial w/\partial x)]}{\partial x} - \frac{\partial u_f}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial[u_s - y(\partial v/\partial x) - z(\partial w/\partial x)]}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{22}$$

where the contribution of u_f has been separated. With $(\partial u_f/\partial x)$ evaluated from (16), Eq. (22) simplifies to

$$\varepsilon_{xx} = \frac{\partial u_s}{\partial x} - y\frac{\partial^2 v}{\partial x^2} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u_s}{\partial x} - y\frac{\partial^2 v}{\partial x^2} - z\frac{\partial^2 w}{\partial x^2}\right)^2 \tag{23}$$

The above is equivalent to Eq. (5) in Ref. 6 and Eq. (4) in Ref. 7, although neither publication includes the intermediate step (22). It is interesting to note that ε_{xx} evaluated with (23) without the second-order term can be derived from the axial strain of Eq. (13) without the $(\partial u/\partial x)^2$ term. The strain energy in Ref. 6 can be constructed by using this linearized strain and it has a standard form employed in the linear theory. Kaza and Kvaternik formulate their strain energy with the strain (23), but drop the resultant fourth-order terms in U.

Because of the differences in the approximations made, Vigneron and Kaza and Kvaternik derive different sets of motion equations. Vigneron obtains linear equations for the elastic variables v and w, while Kaza and Kvaternik present a set of second-order dynamics equations for u_s, v, and w. It is important to observe that the term responsible for the stiffening of the beam derived in Refs. 6 and 7 is of first-order and appears via kinetic energy. This occurs because of the particular form assumed for the axial displacement [Eqs. (15) and (16)]. If, as is done by Likins et al., one does not explicitly separate u_f, then the stiffening appears in the motion equations through the strain energy and is fundamentally a nonlinear term.

To complete this section, we note that Kaza and Kvaternik identify four different approaches for deriving linear or nonlinear equations of motion. They show that all "make use of geometric nonlinear theory of elasticity either implicitly or explicitly." Kaza and Kvaternik also discuss whether foreshortening must be explicitly included in the axial displacement field. They conclude that, although it is not necessary, the approaches where the foreshortening effect is accounted for otherwise require special considerations.

IV. Laskin et al., Simo and Vu-Quoc, Meirovitch, Banerjee et al.

In this section, we discuss some of the more recent formulations that have been proposed to account for the geometric stiffening in multibody systems. These are presented in the chronological order of their publication in open literature.

Laskin et al.

Similar to Vigneron[6] and Kaza and Kvaternik,[7] Laskin et al.[8] partition the axial displacement u_0 into two parts. Rewriting their Eq. (6) as

$$u = u_{qs} + u_t - y\frac{\partial v}{\partial x} - z\frac{\partial w}{\partial x} \tag{24}$$

it can be directly compared to Eq. (15). Laskin et al. refer to u_{qs} (their v_0) as a "quasisteady" component, which reflects the fact this axial displacement is present even when there is no longitudinal vibration. Indeed, it exists even under static loading. It is argued that the arrangement in (24) allows one to consider the "transient" component u_t (v^* in Ref. 8) as an infinitesimally small, displacement, while u_{qs} may be comparatively large. To discretize the deformation field, Laskin et al. use modal expansion similar in form to Eq. (3) but written for the axial displacement u_t rather than u_0 [u_1 in Eq. (3)]. They note that if one discretizes $u_0 = u_{qs} + u_t$, its modal expansion cannot adequately account for both components without having to include a large number of terms in the expansion.

The motion equations of the beam are derived by employing Kane's[1] method. However, Laskin et al. take a different approach from that in Refs. 1 and 4 to derive the generalized elastic forces. They express the internal force **F** as a function of the "spatial stress tensor," taking into account the nonlinear nature of the deformation gradient (see Ref. 17 for details). The resultant expression for **F** is presented in Eq. (21) of Ref. 8 and we note that it implicitly assumes nonlinear strain-displacement relations. The final form of the generalized elastic forces is given in terms of the coordinates u_{qs}, u_t, v, and w. Thus, Laskin et al. must take an intermediate step of substituting for the stresses from stress-strain and then strain-displacement relations. What is most notable about their approach is that, by using the "nonlinear" relation for **F** in terms of the stresses, they derive the geometric stiffening component of elastic forces with a linear form of the strain-displacement relations. The resulting stiffening terms are linear in the generalized elastic coordinates, but they, and the generalized inertia forces, explictly depend on the quasi-steady axial displacement u_{qs}. Accordingly, the final motion equations require this variable as an input.

In applying their dynamics equations, Laskin et al. specify u_{qs} as the stretch that would occur if the beam were executing its nominal or intended motion. This stretch is defined as a solution of a linear differential equation that as can be deduced from the examples given in their paper, is

$$EA\frac{\partial^2 u_{qs}}{\partial t^2} = -\frac{\partial P}{\partial x} \triangleq p \tag{25}$$

For general rotational motions, Laskin et al. propose to approximate the axial load density p by the axial component of the centripetal acceleration (multiplied by an appropriate inertia) of a point on the elastic axis of the beam, i.e.,

$$p = \rho(x + u_{qs})\left(\omega_2^2 + \omega_3^2\right) \tag{26}$$

They suggest that the above should provide a good approximation for the steady-state axial load in the case of rotational motions at low angular acceleration rates. Moreover, it allows for a closed-form solution for u_{qs} that can then be used as input to their dynamics model.

Simo and Vu-Quoc

Simo and Vu-Quoc[9] describe the application of the geometrically nonlinear theory to the dynamic analysis of flexible beams and plates. In contrast to the previous works, with the exception of Ref. 1, they employ an exact kinematic description ("exact" within the plane sections remain plane assumption), not the linearized form described earlier in the paper. This implies an independent treatment of the rotation field, as in Ref. 1. Another significant feature of Simo and Vu-Quoc's formulation is the use of nonlinear strain measures that are invariant under superposed rigid-body motion.

The procedure is illustrated here with an example of a planar beam rotating at a prescribed speed $\Omega(t)$. Then, the position of a generic point in the beam expressed in the floating frame is

$$\boldsymbol{r} + \boldsymbol{u} = \begin{bmatrix} x + u_0 - y\sin\alpha \\ v_0 + y\cos\alpha \end{bmatrix} \tag{27}$$

where α is the angle between the floating x axis and the local normal to the beam cross section. The linearized description of the beam geometry is obtained by assuming $\alpha \ll 1$. Coupled with the assumption of negligible shear deformation, that is $\alpha \approx \partial v_0/\partial x$, this yields the kinematic description employed in Refs. 4–7.

The kinetic energy expression that emanates from the position field (27) is

$$T = \frac{1}{2}\int_0^L \rho\left\{(\dot{u}_0 - \Omega v)^2 + [\dot{v}_0 + \Omega(x + u_0)]^2\right\} \mathrm{d}x + \frac{1}{2}\int_0^L (\dot{\alpha} + \Omega)\frac{\rho I_z}{A}\,\mathrm{d}x \tag{28}$$

The aforementioned nonlinear strain measures for a beam are cited here as

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \kappa \end{bmatrix} = \begin{bmatrix} \cos\alpha\left(1 + \dfrac{\partial u_0}{\partial x}\right) + \sin\alpha\dfrac{\partial v_0}{\partial x} - 1 \\ -\sin\alpha\left(1 + \dfrac{\partial u_0}{\partial x}\right) + \cos\alpha\dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial \alpha}{\partial x} \end{bmatrix} \tag{29}$$

and the corresponding strain energy is

$$U = \frac{1}{2}\int_0^L \left(EA\Gamma_1^2 + GA_s\Gamma_2^2 + EI_z\kappa^2\right)\mathrm{d}S \tag{30}$$

With expressions (28) and (30), Simo and Vu-Quoc use Hamilton's principle to construct the motion equations of a beam. The successive approximations of these equations are derived by approximating the nonlinear strain measures of Eq. (29). Based on linear, second-order and the exact strain measures, Simo and Vu-Quoc present the corresponding linear, third-order, and fully nonlinear equations of motion for the beam. The stiffness term in these equations is expressed in terms of the internal (elastic) forces.

Of particular interest here are the motion equations derived with the second-order approximation for the strain measures. In this case, $\bar{\bar{\Gamma}}_1$ reduces to

$$\bar{\bar{\Gamma}}_1 \approx \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial v_0}{\partial x}\right)^2 \tag{31}$$

which coincides with the approximate Lagrangian strain (for the planar case) of Eq. (10). The strain energy calculated with the above $\bar{\bar{\Gamma}}_1, \bar{\bar{\kappa}} \approx \partial^2 v_0/\partial x^2$, and neglecting the shearing stresses is of the same form as the strain energy expression in Hanagud and Sarkar.[4] Therefore, the third-order motion equations derived in the two references are equivalent.

Finally, Simo and Vu-Quoc develop what they refer to as the consistent linear motion equations. This is carried out by linearizing the third-order equations of motion with the axial internal force P approximated by its zero-order steady-state value. For a beam rotating in a plane, this force is given by Eq. (20). Simo and Vu-Quoc generalize this approximation for a plate rotating at $\omega(t)$ by evaluating the axial load from the lowest-order approximation of the inertia force. It is observed that in the case of a beam rotating at $\omega(t)$, this procedure yields

$$\frac{\partial P}{\partial x} = -p = -\rho x\left(\omega_2^2 + \omega_3^2\right) \tag{32}$$

Comparing the above with expression (26) proposed by Laskin et al. shows that the two expressions are identical to zeroth order.

Meirovitch

In the recent publication, Meirovitch[10] derives an exact set of motion equations for an unconstrained flexible body. They are written in terms of the rigid-body quasicoordinates and the elastic coordinates u, v, and w. The stiffness term in the equations is represented with the homogeneous stiffness differential operators that "include all the terms that can be accounted for in a strain energy function."[10] Meirovitch illustrates his formulation with a system made up of a rigid hub and a beamlike flexible appendage in general translation and rotation. The kinetic energy is derived in the usual manner and contains terms that are of second degree in the elastic variables. The strain energy includes the standard second-order contributions due to bending, as well as the contribution due to "shortening of the projection." The latter is expressed as

$$U_P = \frac{1}{2}\int_0^L \left[\int_x^L p(\zeta, t)\,\mathrm{d}\zeta\right]\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x \tag{33}$$

Meirovitch proposes to determine the force density p from the motion equation for the axial displacement, omitting terms that involve elastic displacements and the control force density. The resultant expression is given by Eq. (29) in Ref. 10, rewritten here as

$$p = \rho\left[-\dot{\mathrm{v}}_1 - \omega_2\mathrm{v}_3 + \omega_3\mathrm{v}_2 + x\left(\omega_2^2 + \omega_3^2\right)\right] \tag{34}$$

Recall that p employed by Laskin et al. to determine the quasi-steady displacement u_{qs} is evaluated from the centripetal component of the inertial acceleration of a mass element on the elastic axis. Meirovitch's expression for p can also be derived from the inertial acceleration distribution, but neglecting the elastic contributions. Thus, the "centripetal component" of p, given by $\rho x(\omega_2^2 + \omega_3^2)$ in Eq. (34), is identical to p of Eq. (26) if one drops u_{qs} in the latter. From (34), one can also conclude that Laskin et al.'s approximation should apply to a beam in general motion if the translational velocity and acceleration are small relative to the angular velocity. Finally, Meirovitch's technique yields the same results as the method used by Simo and Vu-Quoc. Thus, in the case of pure rotation, Eq. (34) reduces to the axial force density of (32).

Banerjee et al.

Similarly to Refs. 4, 5, 9, and 10, the formulation of Banerjee and Dickens[11] is based on the conventional description of the deformation field. The equations of motion are derived by employing the same general methodology as in Refs. 1 and 4. To model the stiffening effect in a flexible body in large rigid-body motion, Banerjee and Dickens introduce the notion of "motion stiffness" as a special case of the geometric stiffness caused by the inertia loading on the body. As noted by Banerjee and Lemak,[12] this motion-induced stiffness has its origin in the strain energy term

$$U_{\mathrm{NL}} = \int_V \varepsilon_{\mathrm{NL}}^T \boldsymbol{\sigma}_0\,\mathrm{d}V \tag{35}$$

where $\boldsymbol{\sigma}_0$ is defined as the "reference" state of stress in the body,[12] whereas $\varepsilon_{\mathrm{NL}}$ collects the nonlinear components of the Lagrangian strains. Thus, one can view U_{NL} as the nonlinear part of U, since it arises from the nonlinear terms in the strain-displacement relations. The key to the method proposed in Ref. 11 is the observation that the stress state $\boldsymbol{\sigma}_0$ results from the inertia loads.

A detailed description of the approach proposed by Banerjee and Dickens is given in Ref. 17. There, it is noted that the nonlinear strain energy of Eq. (35) accounts for the same geometric nonlinearity as that treated in the previous works in the context of a flexible beam. The author also demonstrates that Banerjee and Dickens' expression for the axial inertia force is equivalent to Meirovitch's axial component of the internal force density. Finally, it is pointed out that the procedure in Ref. 11 is based on a fundamental approximation. It is that the stresses which contribute to the stiffening of the body are only those that are caused by the inertia loads. The author believes that the consistent formulation of the strain energy in Eq. (35) requires the complete state of stress in the body. While the approximation for $\boldsymbol{\sigma}_0$ suggested in Ref. 11 is valid for a single body in the absence of external forces, it is not appropriate for multibody systems in which each body is acted upon by interbody forces.

V. Discussion

The review presented suggests that to account for the geometric stiffening of an elastic body undergoing large motions, it is necessary to introduce the geometric nonlinearity into the model. This is done explicitly or implicitly through the use of Lagrangian strains or, stated differently, the nonlinear strain-displacement relations. These account for the coupling between the transverse and axial deformations, which indeed is the cause of stiffening in a deformable body.

There are, however, different approaches to incorporate the geometric stiffening effect into the dynamics equations. To distinguish between them, the author proposes a classification according to the following two criteria: 1) kinematic description of the deformation field (criterion DF) and 2) the formulation of the strain energy function (criterion SE). It is felt that these represent two most general criteria that can fully characterize a particular approach. The first one is related to the assumed displacement field, which in turn determines the generalized coordinates employed in deriving the motion equations. The second criterion defines the form of the elastic force in the motion equations. According to each criterion, the following possible cases are identified.

Criterion DF. The deformation field can be described by any of the following sets of variables:

DF-1. Three independent elastic translations $\{u, v, w\}$ (Refs. 4, 5, and 9–12): This set is the accepted choice used in linear and nonlinear theories of elasticity, as well as structural analysis. It requires no a priori decision on the foreshortening of the body and in that sense is general. Contrary to what has been previously stated, the independence of u, v, and w does not preclude the fact that, in case of a beam, part of u is dependent on v and w. Neither is it necessary to separate u into two components, nor to explicitly account for the coupling between it and the transverse displacements. It should also be observed[17] that discretizing u with a standard expansion does not imply "premature linearization." As exemplified by the formulations of Likins et al. and Hanagud and Sarkar, geometric stiffening can be modeled with this option, provided one incorporates the nonlinear strain-displacement relations in the evaluation of elastic forces:

DF-2. Three independent elastic translations $\{u_s, v, w\}$ (Refs. 6–8): In this case, the foreshortening u_f is explicitly separated from u_s in the axial displacement field. It may be specified in terms of v and w or left as a parameter to be defined by the user. Either case, however, involves making an approximation for u_f, although in the former it is quantified a priori. As well, the concept of "foreshortening" needs to be generalized for bodies other than beams.

DF-3. Three independent but nonorthogonal elastic deformations $\{s, v, w\}$ (Ref. 1): Employment of the stretch variable s is uncommon and provides, at the least, an interesting alternative to the conventional approach. However, the author does not believe that it must be used to formulate the dynamics equations in order to capture the stiffening of the beam. The main advantage of using this variable is that the strain energy retains its "linear" form. But, the kinetic energy is more complex than those obtained with the DF-1 and DF-2 descriptions. As with the DF-2 option, the notion of "stretch" requires appropriate generalization for other types of elastic bodies.

Criterion SE. Two basic formulations of the strain energy can be distinguished:

SE-1. The total strain energy and the corresponding stiffness term are formulated as a function of nonlinear strains and subsequently, the displacements. This formulation, employed in Refs. 1 and 4–7 can be viewed as a displacement formulation. Its main advantage for any description of the deformation field is that one is not required to make any additional assumptions or approximations. In particular, one does not have to ponder on what is the state of stress in the body and how to compute it. Furthermore, since the motion equations are traditionally formulated in terms of displacements, this approach fits naturally into the framework of dynamics simulation.

SE-2. The strain energy is subdivided into a classic linear contribution and a nonlinear part, where the latter is constructed from the stresses or internal forces in the body and the nonlinear components of the Lagrangian strains.[10–12] This formulation can be considered as a force formulation. Its drawback is that it requires an approximation for the stress field in the body and hence, in contrast to SE-1, is inherently approximate.

As demonstrated in the previous sections, it is a particular combination of the coordinates and the strain energy formulation that determines how the geometric stiffening is incorporated into the motion equations as well as in what form it appears. It is therefore appropriate to comment on the two strain energy formulations in combination with different displacement fields.

SE-1/DF-1. In this approach, taken by Hanagud and Sarkar,[4] the stiffening term appears in the motion equations through the strain energy and is a nonlinear function of elastic coordinates. The method corresponds to the third approach identified by Kaza and Kvaternik[7] and, as they comment, is "the one usually employed for a general three-dimensional rotating body." However, contrary to Kaza and Kvaternik's conclusion, we believe that it does not require special consideration but is the most general and accurate. Moreover, this approach can be employed to extend the existing "small" deflection dynamics formulations to incorporate "large" deflection theories.

SE-1/DF-2. With this formulation, the foreshortening term is always present in the kinetic energy expression and may or may not appear in the strain energy. In the simplest case, as in Vigneron's[6] formulation, the resultant stiffening term takes a linear form. From the computational standpoint, this method of modeling stiffening allows one the option of dropping the axial deformation u_s from the model—not a trivial advantage, as it considerably improves the efficiency of the numerical integration. The method, however, introduces additional error because of the approximation made in assuming a particular form for u_f.

SE-1/DF-3. This approach produces a linear geometric stiffening term via kinetic energy (or generalized inertia force). Compared to the SE-1/DF-1 option, it can be just as accurate but not as general.

Among the works presented in this paper, the SE-2 formulation has only been used with the $\{u, v, w\}$ (DF-1) description of the deformation field. In this case, the geometric stiffening term results from the "nonlinear" strain energy. With appropriate assumptions, it is linear in elastic variables but involves rigid-body accelerations and velocities. Thus, the dynamics equations derived with the SE-2/DF-1 approach no longer have a symmetric mass matrix, which makes evaluation of accelerations more costly.

To summarize, we believe that a description of the deformation field in terms of u, v, and w combined with the displacement formulation of the strain energy is the most general approach. It does not require an approximation of foreshortening or the stress state of the body—two critical advantages for applications to multibody systems. A definitive statement on the relative efficiency of the different procedures can only be made through implementation of all methods. The following section contains some results in evidence of their numerical performance.

VI. Numerical Comparison

This section presents a series of numerical results for a benchmark problem—a flexible beam undergoing a spin-up maneuver. Although this system has been considered in a number of publications, there is not a single beam example for which all four approaches identified in the previous section can be compared from the published results. Therefore, the following procedure is adopted towards that goal. First, a detailed numerical comparison of two approaches, SE-1/DF-1 and SE-1/DF-2, is made. Both of these have been implemented by the author, which allows us to comment on the more subtle issues, such as ease of implementation and convergence of the discretization scheme. As well, in implementing the SE-1/DF-2 approach, inertial terms up to third order are included in the motion equations and their significance is assessed. This in itself represents a novel contribution. The second part of the comparison makes use of a different flexible-beam system, one for which results for the SE-1/DF-3 and SE-2/DF-1 formulations are available in literature. A comparison of responses predicted by all four approaches is then made for this beam with the results generated by the author and those cited from the literature.

SE-1/DF-1 vs SE-1/DF-2

Equations of Motion and Discretization

The continuous dynamics equations for a flexible beam rotating about the z axis at a prescribed speed $\Omega(t)$ are presented below. The equations obtained with both formulations are based on the approximate nonlinear strain of Eq. (31). With the exception of the rotary inertia terms, all terms resulting from the respective kinematics descriptions [Eqs. (14) and (15)] are retained.

SF-1/DF-1 dynamics equations:

$$\rho\left[\ddot{u}_0 - \Omega^2 x - \left(\Omega^2 u_0 + 2\Omega\dot{v} + v\dot{\Omega}\right)\right] + EAu_0'' - EAv'v'' = 0$$

$$\rho\left[\ddot{v} + \dot{\Omega}x - \left(\Omega^2 v - 2\Omega\dot{u}_0 - \dot{\Omega}u_0\right)\right] + EI_z v'''' - EA\left[v'u_0'' + u_0'v'' + \tfrac{3}{2}(v')^2 v''\right] = 0 \qquad (36)$$

SE-1/DF-2 dynamics equations:

$$\rho\left[\ddot{u}_s - \ddot{u}_f - \Omega^2 x - \left(\Omega^2 u_s + 2\Omega\dot{v} + v\dot{\Omega}\right) + \underline{\Omega^2 u_f}\right] + AEu_s'' = 0$$

$$\rho\left[\ddot{v} + \dot{\Omega}x - \left(\Omega^2 v - 2\Omega\dot{u}_s - \dot{\Omega}u_s\right) - \underline{(2\Omega\dot{u}_f - \dot{\Omega}u_f)} - (Tv')'\right] + EI_z v'''' = 0 \qquad (37)$$

where

$$T = \int_x^L T_I \, \mathrm{d}x$$

$$T_I = -\ddot{u}_s + \underline{\underline{\ddot{u}_f}} + [\Omega^2 x] + \left(\Omega^2 u_s + 2\Omega\dot{v} + v\dot{\Omega}\right) - \underline{\underline{\underline{\Omega^2 u_f}}} \qquad (38)$$

The $(\cdot)'$ notation in the above denotes partial differentiation $\partial/\partial x$; different terms have been grouped and listed according to their order. The last terms in Eqs. (36) are of second and third order and are derived from the third- and fourth-order terms in the strain energy. The markings of the terms in Eqs. (37) and (38) refer to their counterparts in the kinetic energy of Eq. (21), where recall the term in the square brackets is responsible for the stiffening of the beam. It is noted that Eqs. (37) are closest to the dynamics equations derived in Ref. 7. They differ by the third-order inertial terms (omitted in that reference), but do not contain terms of third order in the spatial derivatives of the elastic displacements. Lastly, the present equations accommodate motions at a nonconstant angular speed.

In our implementation, both sets of equations are discretized according to the Galerkin procedure with the usual shape functions for a beam element. The discretization was carried out with Maple symbolic computation program. Below, we remark on the structure of the resulting discretized equations for the two formulations.

In the SE-1/DF-1 approach, the discrete equations have a standard form, but with the constant stiffness matrix replaced by a configuration-dependent stiffness matrix. If the beam is discretized with more than one element, the latter can be assembled using the established finite element assembly procedure. Banerjee and Lemak[12] pointed out that evaluation of the stiffness term in the SE-1/DF-1 formulation requires that this stiffness matrix be updated at each time step in the simulation, which may be computationally costly. This is supported by our numerical results. The discretization of the continuous equations resulting from SE-1/DF-2 approach is relatively simple if one retains only the linear stiffening term. If higher order terms are included, then the process becomes more complicated and would not even be feasible without symbolic manipulation software. Moreover, the time derivative terms in T_I as well as the $\ddot{u}_f$-term in the axial motion equation modify the usual symmetric structure of the mass matrix, which is also configuration dependent. The assembly of this matrix requires a special procedure and adds to the computational cost.

Approximate Models

Numerical results are presented here for several approximations of Eqs. (36) and (37). The two simpler approximate models are common to both formulations. In particular, the first of these does not contain any inertial terms that involve elastic displacements or their rates and has been coined by Padilla and von Flotow[15] as ruthlessly linearized (RL). The second model includes terms that are linear in elastic coordinates with the exception of the linear stiffening term in the SE-1/DF-2 formulation. The response of the beam predicted by this model is unstable and is not presented here. Further approximations must be considered separately for the two formulations.

One approximation that has been used by some researchers[16] in conjunction with the SE-1/DF-1 formulation is obtained by retaining only the third-order nonlinear contribution to the strain energy and treating the EAu_0' as "constant" when deriving the corresponding stiffness term.[17] The resulting model will be referred to as second order and denoted by S-2, the letter S to signify that the nonlinearity appears in the stiffness term. The last model considered for the SE-1/DF-1 approach contains all terms in Eq. (36) and is of third order (S-3).

For the dynamics equations derived with the SE-1/DF-2 approach, one can identify the following approximations. The consistently linearized model (I-1), as the name suggests, is a linear approximation of Eqs. (37) and it includes the stiffening term. The consistent second-order model (I-2) is derived from the third-order expression for the kinetic energy. It is defined with T_I of Eq. (38) without the underlined terms. The final model (I-3) includes all terms in Eqs. (37). In addition to these, an approximate linear model (I-1a) is defined, which is similar to I-1 but excludes the $\ddot{u}_f$-term in the axial equation. This model generates results indistinguishable from the I-1 approximation but is more efficient.

To compare the performance of different models, a flexible-beam example of Ref. 4 is used. Figure 1a shows the transverse deflection v of the beam obtained with one-element discretization for the RL, S-2, and S-3 models. The S-2 response has been compared to the response presented in Ref. 16 (for a beam with slightly different properties.) The maximum transverse deflection shown in that reference is -0.59 m, whereas the corresponding value predicted with the present implementation is -0.57 m. Hanagud and Sarkar,[4]

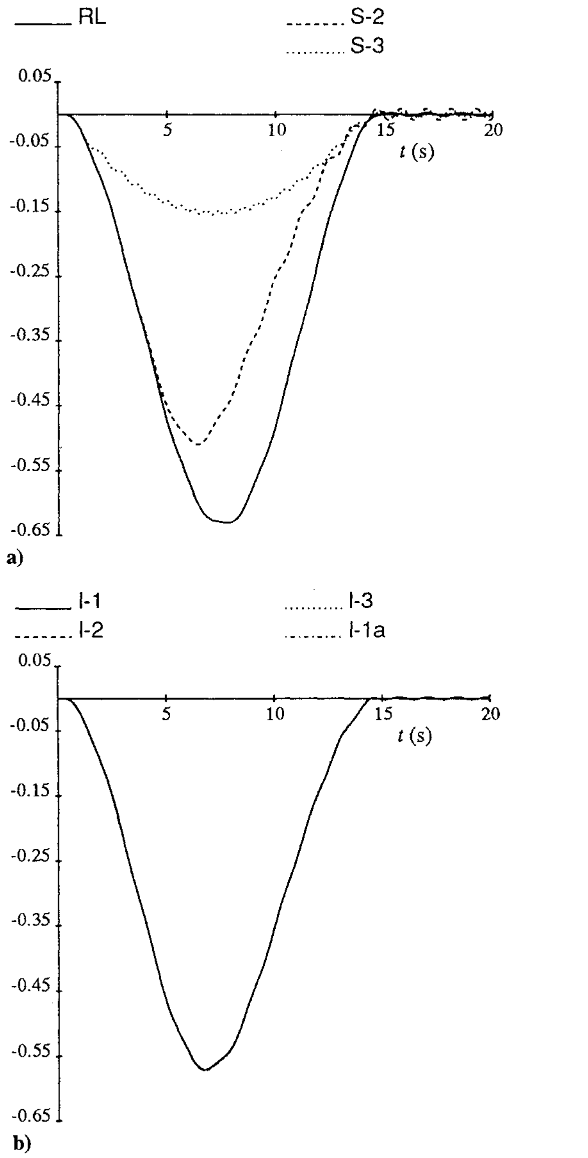

Fig. 1 Tip displacement v (meters) for SE-1/DF-1 and SE-1/DF-2 models.

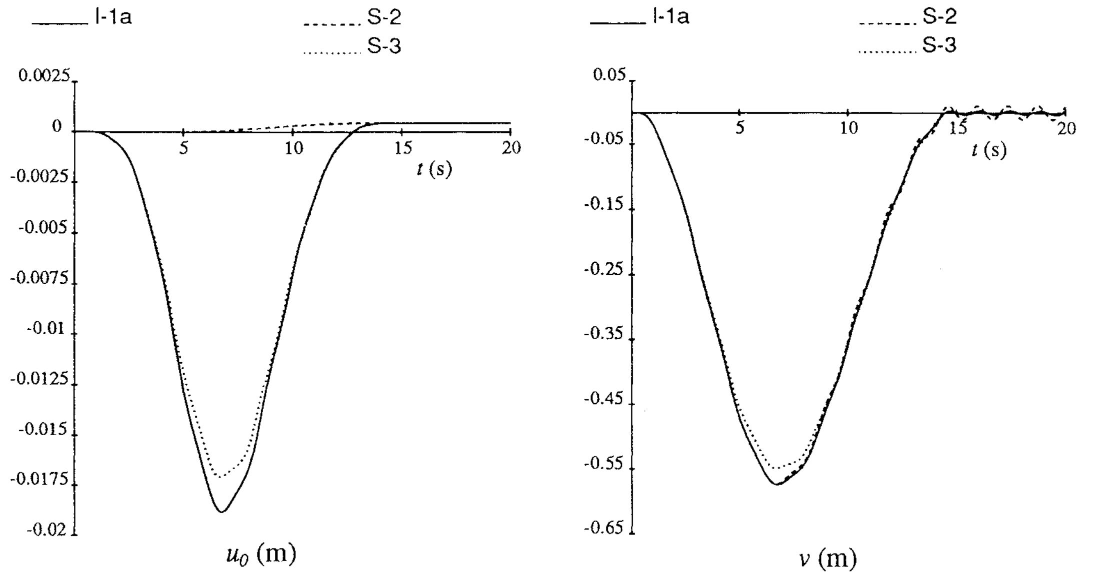

Fig. 2 Axial and transverse tip displacements for different models.

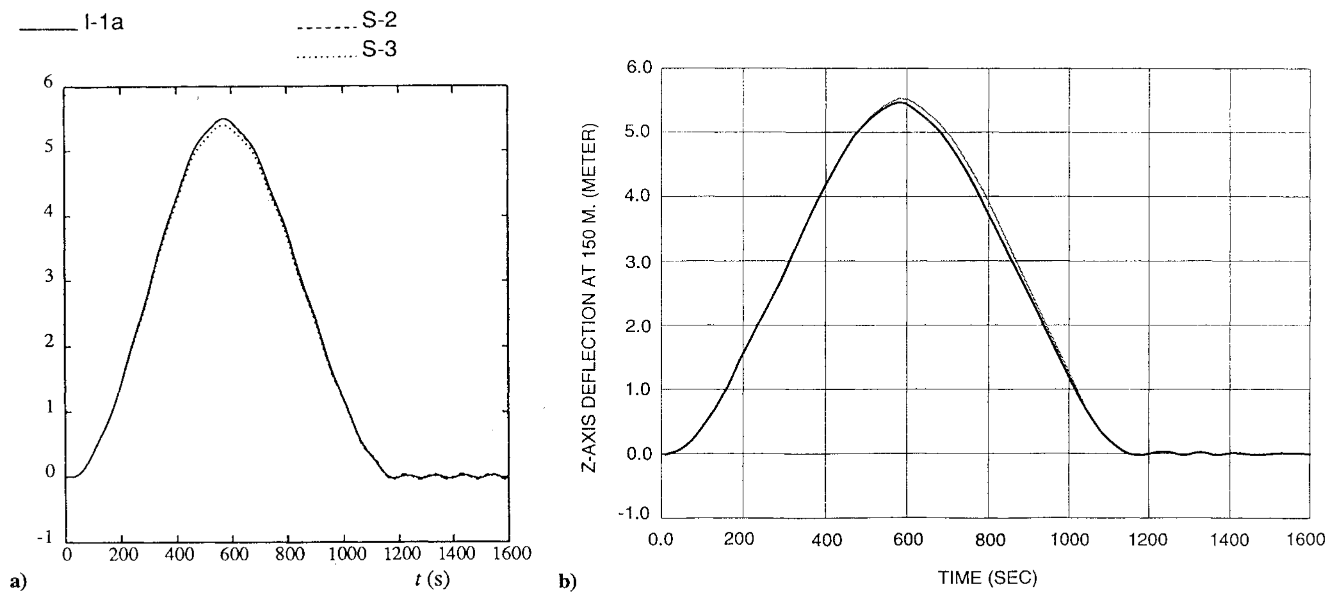

Fig. 3 Comparison of different approaches.

present the axial and transverse displacements obtained with the S-3 model for a beam discretized with one axial and three transverse eigenfunctions. The results of Fig. 1a are in excellent agreement with theirs. As noted below, however, the solution predicted with one-element discretization of the S-3 model greatly underestimates the deformation of the beam. Figure 1b displays the responses for the I-1, I-2, I-3, and I-1a models defined for the SE-1/DF-2 dynamics equations. Models I-1 and I-1a produce identical results and the second- and third-order inertial terms introduce indistinguishable difference in the response. It is noted that the transverse deflection of Fig. 1b is identical to that presented in Ref. 9.

Another aspect of the two formulations investigated by the author is convergence of the finite element discretization scheme. As shown in Ref. 17 for the present beam example, the I-1a, S-2 and S-3 models converge in two, four and over 30 elements, respectively! Thus, the SE-1/DF-2 formulation yields a much better performance in terms of the convergence of the discretization scheme. By contrast, the exact SE-1/DF-1 approach has very poor convergence characteristics. This supports the remarks made by Laskin et al.[8] to the effect that a large number of modes is required if one does not explicitly separate the stretch from the foreshortening of the beam. We suggest that convergence of the SE-1/DF-1 formulation can be improved by using higher-order shape functions to discretize the axial displacement in the beam.

To summarize the comparison of the responses, the converged axial and transverse displacements are plotted in Fig. 2. It is evident that the results predicted by different formulations are in good agreement. It is also noted that for a beam spin-up problem, the S-2 approximation of the SE-1/DF-1 approach provides a very accurate solution.

It is fitting to comment on the efficiency of the computation of the various models. The CPU times and the number of function evaluations incurred during the numerical integration for responses of Fig. 1 are summarized in Table 1. Also included are the data for the I-1a model without the axial deformation. These results demonstrate the superior efficiency of the SE-1/DF-2 formulation, particularly due to the fact that it allows one to drop the axial motion equation. They are also in support of the previous observation regarding the high cost associated with the assembly of the configuration-dependent stiffness or mass matrices. The differences in the computational performance of the two formulations become particularly pronounced when one compares the corresponding data for the converged solutions.

Comparison with SE-1/DF-3 and SE-2/DF-1 Approaches

The flexible beam considered here is a 150-m-long antenna tube of Ref. 12. Figure 3a contains the converged transverse deflection at the tip calculated with the S-2, S-3, and 1-1a models defined

Table 1 Performance statistics for different models

Model	CPU on HP 730, s	Number of function evaluations
RL	5.0	46,091
S-2	7.0	47,068
S-3	10.0	46,221
I-1	10.0	46,814
I-1a	7.0	47,069
I-1a, without u_s	0.3	1,838

for the SE-1/DF-1 and SE-1/DF-2 formulations. Figure 3b displays the corresponding response reproduced from Ref. 12. It contains the deflection predicted with the formulation proposed in that reference (thick line) and the response obtained with the theory developed in Ref. 1 (lighter line). These were classified here as SE-2/DF-1 and SE-1/DF-3 approaches. It can be seen that the responses obtained with the SE-1/DF-1 and SE-1/DF-2 formulations are close to the results of Fig. 3b.

VII. Concluding Remarks

This manuscript presented an exposition of several approaches to model the geometric stiffening effect for dynamics simulation of flexible-body systems. The survey included papers published in the period from 1973 to 1991. Although it does not represent a complete literature survey, it covers a wide range of formulations developed for the problem.

In reviewing these works, it was demonstrated that all formulations incorporate the geometric nonlinearity, which in case of beam-like bodies gives rise to foreshortening. Two key characteristics of the different methods were identified, based on which a general classification was put forward. The author has also established the interrelationships between the various approaches, provided a number of clarifications, and offered opinions on their benefits. Numerical example for a flexible beam brought to light the differences in their performance. It is hoped that this work will contribute to a better understanding of the origin of geometric stiffening and how this effect can be incorporated into the dynamics model of a flexible body undergoing large rigid-body motion.

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